

המשפט של קושי (Cauchy's Theorem) 5

$$\int_0^{2\pi} \frac{dt}{1-2a \cos t + a^2} = \int_0^{2\pi} \frac{dt}{1+a^2-a(e^{it}+e^{-it})} \cdot \frac{ie^{it}}{ie^{it}} = \quad (1)$$

$$= -i \int_0^{2\pi} \frac{ie^{it} dt}{(1+a^2)e^{it} - a(e^{2it}+1)} = \left\{ \begin{array}{l} \gamma: [0, 2\pi] \rightarrow \mathbb{C} \\ \gamma(t) = e^{it} \end{array} \right\} =$$

$$= -i \int_{\gamma} \frac{dz}{(1+a^2)z - a(z^2+1)} = i \int_{\gamma} \frac{dz}{az^2 - (1+a^2)z + a} =$$

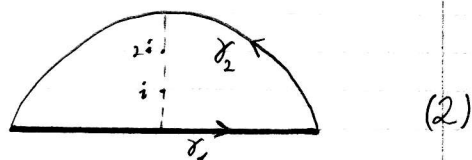
$$= i \int_{\gamma} \frac{dz}{(az-1)(z-a)} = i \int_{\gamma} \frac{1}{1-a^2} \left( \frac{a}{az-1} - \frac{1}{z-a} \right) dz =$$

$$= \frac{i}{1-a^2} \left( \int_{\gamma} \frac{dz}{z-\frac{1}{a}} - \int_{\gamma} \frac{dz}{z-a} \right) = \frac{i \cdot 2\pi i}{1-a^2} \cdot (n(\gamma, \frac{1}{a}) - n(\gamma, a)) =$$

$$\left[ \text{מכיוון } |a| < 1 \Rightarrow n(\gamma, a) = 1, n(\gamma, \frac{1}{a}) = 0 \left( \left| \frac{1}{a} \right| = \frac{1}{|a|} > 1 \right) \right]$$

$$= -\frac{2\pi}{1-a^2} (0-1) = \underline{\underline{\frac{2\pi}{1-a^2}}}$$

$$\gamma_1(t) = t, \gamma_2(t) = Re^{it}$$



$$\int_{\gamma} \frac{dz}{(z^2+1)(z^2+4)} = \frac{1}{3} \int_{\gamma} \left( \frac{1}{z^2+1} - \frac{1}{z^2+4} \right) dz = \frac{1}{3} \frac{1}{2i} \int_{\gamma} \left( \frac{1}{z-i} - \frac{1}{z+i} \right) dz -$$

$$- \frac{1}{3} \cdot \frac{1}{4i} \int_{\gamma} \left( \frac{1}{z-2i} - \frac{1}{z+2i} \right) dz = -\frac{i}{6} \cdot 2\pi i (n(\gamma, i) - n(\gamma, -i)) +$$

$$+ \frac{i}{12} \cdot 2\pi i (n(\gamma, 2i) - n(\gamma, -2i)) = \frac{\pi}{3} (1-0) - \frac{\pi}{6} (1-0) = \underline{\underline{\frac{\pi}{6}}}$$

$$\left| \int_{\gamma_2} \frac{dz}{(z^2+1)(z^2+4)} \right| = \left| \int_{\gamma_2} \frac{dz}{z^4+5z^2+4} \right| = \left| \int_0^{\pi} \frac{iRe^{it} dt}{R^4e^{4it}+5R^2e^{2it}+4} \right| \leq$$

$$\leq \int_0^{\pi} \frac{R dt}{|R^4e^{4it}+5R^2e^{2it}+4|} \stackrel{(*)}{\leq} \int_0^{\pi} \frac{R dt}{R^4-5R^2-4} = \frac{\pi R}{R^4-5R^2-4} \xrightarrow{R \rightarrow \infty} 0$$

$$(*) : |R^4e^{4it}+5R^2e^{2it}+4| \geq |R^4e^{4it}| - |5R^2e^{2it}| - 4 =$$

$$= R^4 - 5R^2 - 4 \quad \text{עבור } R \text{ מספיק גדול}$$

$$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)(x^2+4)} = \int_{\gamma_1} \frac{dz}{(z^2+1)(z^2+4)} = \int_{\gamma} f dz - \int_{\gamma_2} f dz = \frac{\pi}{6} - 0 = \underline{\underline{\frac{\pi}{6}}}$$

$$\int_{\gamma} \frac{e^{iz}}{z^2+a^2} dz = \frac{1}{2ia} \left( \int_{\gamma} \frac{e^{iz}}{z-ia} dz - \int_{\gamma} \frac{e^{iz}}{z+ia} dz \right) \quad (3)$$

$$\frac{e^{iz}}{z+ia} \quad \text{לפי } -ia \text{ נקודה בתוך } \gamma \text{ מסלול סגור}$$

$$\int_{\gamma} \frac{e^{iz}}{z+ia} dz = 0 \quad \text{אם } \gamma \text{ סגור וחסום}$$

$$\int_{\gamma} \frac{e^{iz}}{z^2+a^2} dz = \frac{1}{2ia} \int_{\gamma} \frac{e^{iz}}{z-ia} dz \quad \text{לפי}$$

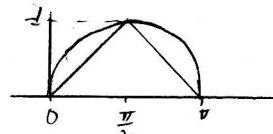
$$e^{iz} = 1 + \sum_{n=0}^{\infty} \frac{(iz)^n}{n!} = \dots \quad e^{i(z-ia)} = 1 + \sum_{n=0}^{\infty} \frac{i^n}{n!} (z-ia)^n$$

$$\text{, and by } \frac{e^{i(z-ia)} - 1}{z-ia} = \frac{\sum_{n=1}^{\infty} \frac{i^n}{n!} (z-ia)^n}{z-ia} = \sum_{n=1}^{\infty} \frac{i^n}{n!} (z-ia)^{n-1} \quad \text{p.s.}$$

$$\int_{\gamma} \frac{e^{iz+a} - 1}{z-ia} dz = 0 \quad \text{p.s. and by } \frac{e^{iz+a} - 1}{z-ia} \quad \text{and so}$$

$$\begin{aligned} \int_{\gamma} \frac{e^{iz}}{z^2 + a^2} dz &= \frac{1}{2ia} \int_{\gamma} \frac{e^{iz}}{z-ia} dz = \frac{e^{-a}}{2ia} \left( \int_{\gamma} \frac{e^{iz+a} - 1}{z-ia} dz + \int_{\gamma} \frac{dz}{z-ia} \right) \\ &= \frac{e^{-a}}{2ia} \int_{\gamma} \frac{dz}{z-ia} = \frac{e^{-a}}{2ia} \cdot 2\pi i \cdot n(1, ia) = \frac{\pi}{ae^a} \end{aligned}$$

$$\begin{aligned} \left| \int_{\gamma_2} \frac{e^{iz}}{z^2 + a^2} dz \right| &= \left| \int_0^{\pi} \frac{e^{iRe^{it}}}{R^2 e^{2it} + a^2} \cdot iRe^{it} dt \right| \leq \int_0^{\pi} \frac{R \cdot e^{-R \sin t}}{|R^2 e^{2it} + a^2|} dt \leq \\ &\leq \int_0^{\pi} \frac{R e^{-R \sin t}}{R^2 - a^2} dt \quad (\leq) \end{aligned}$$



$$\text{p.s. } \sin x \geq 2 - \frac{2}{\pi} x, \quad \frac{\pi}{2} < x \leq \pi \quad \text{and} \quad \sin x \geq \frac{2}{\pi} x, \quad 0 \leq x \leq \frac{\pi}{2} \quad \text{p.s.}$$

$$\begin{aligned} (\leq) \frac{R}{R^2 - a^2} \left( \int_0^{\frac{\pi}{2}} e^{-\frac{2R}{\pi} t} dt + \int_{\frac{\pi}{2}}^{\pi} e^{-R(2 - \frac{2}{\pi} t)} dt \right) &= \\ = \frac{R}{R^2 - a^2} \left[ -\frac{\pi}{2R} (e^{-R} - 1) + e^{-2R} \cdot \frac{\pi}{2R} (e^{2R} - e^R) \right] &= \\ = \frac{\pi}{R^2 - a^2} \cdot \frac{1}{2} (-e^{-R} + 1 + 1 - e^{-R}) = \frac{\pi(1 - e^{-R})}{R^2 - a^2} \xrightarrow{R \rightarrow \infty} 0 \end{aligned}$$

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\cos x}{x^2 + a^2} dx &= \int_{\gamma_1} \frac{\cos z}{z^2 + a^2} dz = \operatorname{Re} \int_{\gamma_1} \frac{e^{iz}}{z^2 + a^2} dz = \\ &= \operatorname{Re} \left( \int_{\gamma} - \int_{\gamma_2} \right) = \operatorname{Re} \left( \frac{\pi}{ae^a} - 0 \right) = \frac{\pi}{ae^a} \end{aligned}$$

$$\begin{aligned} \int_{-\infty}^{\infty} \frac{\cos x}{(x^2 + a^2)(x^2 + b^2)} dx &= \frac{1}{a^2 - b^2} \int_{-\infty}^{\infty} \left( \frac{1}{x^2 + b^2} - \frac{1}{x^2 + a^2} \right) \cos x dx \quad (1) \\ \text{p.s. } \frac{1}{a^2 - b^2} \cdot \left( \frac{\pi}{be^b} - \frac{\pi}{ae^a} \right) \end{aligned}$$